A Simple Introduction to the Laplace Equation

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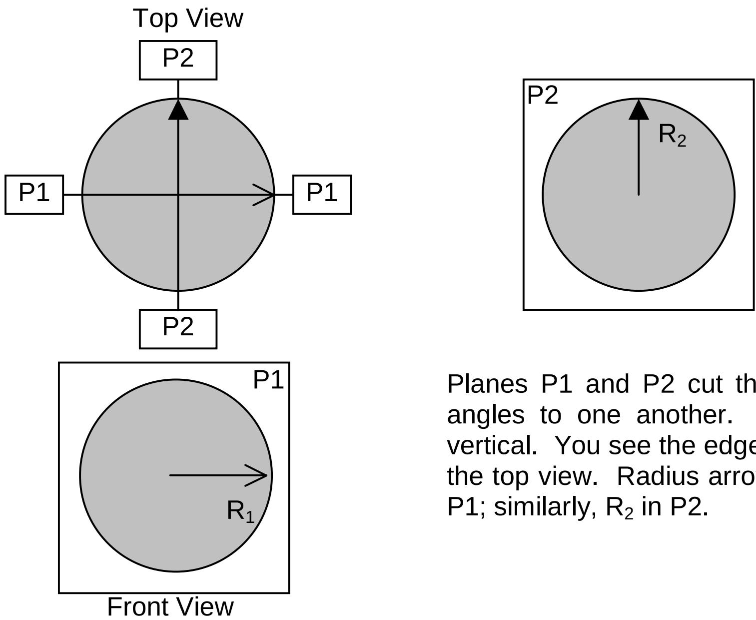
The Laplace equation relates pressure to drop curvature and is fundamental to all drop shape analysis methods. This note will attempt to explain its meaning in a simple way. First, imagine the simplest possible droplet of liquid: a small sphere. We want to define the radii of curvature in a precise way, so consider the following engineering drawing of the drop.



Planes P1 and P2 cut the sphere at right angles to one another. Both planes are vertical. You see the edges of the planes in the top view. Radius arrow R_1 lies in plane P1; similarly, R_2 in P2.

The first important concept is that of radius of curvature: this says that if a sphere were fitted to the surface at a point, then the radius of the sphere is the radius of curvature for that point on the surface. For the spherical drop, this is trivial: the radius of curvature is the radius of the sphere, but for more complex surfaces finding the radius of curvature is difficult.

Secondly, we have two planes cutting the spherical drop. Let P1 be the first plane. It doesn't matter exactly where P1 is placed as long as it passes through the center of the sphere. P2 is then placed perpendicularly to P1, or *normal* to P1. In the case of the sphere, this is again trivial because the sphere is symmetric in all directions, so R1 and R2 will be the same no matter where P1 and P2 are placed (but both must pass through the center). For more complex surfaces, the equations for the planes and their radii of curvature will be simpler if the planes are chosen "wisely", but the concepts are the same: two planes and two radii of curvature. Only in the case of the sphere are R1 = R2 always and = radius of sphere. In summary, any point on a curved surface can be defined by its two *principal* (i.e., normal) radii of curvature. Notice a flat surface has infinite radii of curvature.

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Laplace observed there must be a pressure difference associated with the curvature of the surface. This pressure difference is between the inside and the outside of the drop. Imagine traveling along one of the radii of curvature arrows (vectors). While inside the drop, the hydrostatic pressure is, say, P_1, but when outside, beyond the surface, the pressure is P_0. The pressure difference is caused by the attraction of the liquid molecules to each other and is otherwise known as surface tension γ. (Do not confuse pressure P_1 with plane P1.)

$$P_1 - P_0 = \Delta P = \gamma\,(1/R_1 + 1/R_2)$$

For the simple case of the sphere,

$$\Delta P = 2\gamma / R$$

and for an absolutely flat surface, because the radii of curvature are then infinite,

$$\Delta P = 0.$$

Finally, consider the case of a tall drop with gravity present. A hanging, or pendant, drop is a good example. The pressure in the drop must change as one goes up or down vertically in the drop, because of gravity. With the difference in densities between inside and outside the drop called ρ ("rho"), acceleration due to gravity called g, and height in the drop h, the *change* in pressure with height is

$$\Delta P = \rho\, g\, h$$

which is nothing more than ordinary hydrostatic pressure. Note the density of water at sea level is essentially 1g/cc and air is 0.0011g/cc. The two equations for pressure are put together in the Laplace-Young (or Young and Laplace) equation:

$$\Delta P = \gamma\,(1/R_1 + 1/R_2) = \rho\, g\, h$$

This is the equation by which surface tension is extracted from shape of either pendant or sessile (sitting) drops. The solution is difficult because the radii of curvature are complex expressions for real drop shapes, but there are numerical methods which approximate the solution with more than sufficient accuracy.

Symbol	Meaning	How Obtained
R_1	Radius of curvature #1	From drop profile at any point P
R_2	Radius of curvature #2	From drop profile at same point P
h	Height in drop	Height of P from drop apex ("tip")
g	Acceleration due to gravity	Handbook
ρ	Density difference of materials	Handbook
γ	Surface or Interfacial Tension	By solution of equation

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